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ORIGINAL RESEARCH

Enforcing fair cooperation in production-inventory settings with heterogeneous agents

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Abstract
Production-Inventory settings with heterogeneous agents appear frequently in the study of supply chain management. For instance, there are Production-Inventory situations in which certain agents are essential as they can reduce the costs of other agents (followers) when they cooperate with each other. The study of such a cooperation can be modelled by means of a cooperative game and studied finding fair cost allocations. These class of cooperative games was introduced in Guardiola et al. (in Games Econ Behav 65:205–219, 2009) where it was also proposed the Owen point. This cost allocation is an appealing solution concept that for Production-Inventory games (PI-games) is always stable, in the sense of the core. The Owen point allows all the players in the game to operate at minimum cost but it does not take into account the cost reduction induced by essential players over their followers. Thus, it may be seen as an altruistic allocation for essential players what can be criticized. The aim of this paper is two-fold: to introduce new core allocations for PI-games improving the weaknesses of the Owen point and to study the structure and complexity of set of stable cost allocations (the core) of PI-games.

Keywords Cooperative game · Core · Cost allocation · Production-inventoy setting

Mathematics Subject Classification 91A12 · 90B05

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1 Introduction

Classical models in Supply Chain Management assume that a single agent aims to maximize his utility, by developing, individually, adequate mathematical optimization models and providing suitable solution procedures. That is, supply chain models support decision-makers to achieve their goals optimally without any interaction among them. Nevertheless, in nowadays complex world where, in most cases, several heterogeneous agents interact, this simplifying assumption of a single agent is not suitable. In these setting, apart from the pure utility maximization problem, decision-makers often face new problems that come out either from cooperation or competition among the agents. A tutorial on game theory in supply chain analysis is Cachon and Netessine (2004). The authors discuss both noncooperative and cooperative game theory in static and dynamic situations. among retailers and the supplier. A few years before, Borm et al. (2001) surveys the research area of cooperative games associated with several types of operations research problems in which various decision makers, interested on minimizing and allocating total join costs, are involved. These agents face to the problem of how to distribute these joint costs back to the individual decision maker. The paper is surveyed on the basis of a distinction between the nature of the underlying optimisation problem: connection, routing, scheduling, production and inventory. A more recent review, Fiestras-Janeiro et al. (2011) focus on the applications of cooperative game theory in the management of centralized inventory systems.

This analysis of cooperation and profit allocation in supply chain management has given rise to a fruitful literature among which we highlight the following papers. Guardiola et al. (2007) focus on the coordination of actions and the allocation of profit in supply chains under decentralized control in which a single distributor supplies several retailers with goods for replenishment of stocks. The goal of all the agents is to maximize their individual profits. Since the outcome under decentralized control is inefficient, cooperation among retailers and the supplier, by means of coordination of actions, may improve the individual profits. By using cooperative game theory, the authors propose a tailor-made stable, in the sense of the core, allocation of the joint profit. Later, Drechsel and Kimms (2011) study how to allocate cost of the cooperation on the capacitated lot sizing problem in a stable (core) and fair way. They find there are hard-to-solve optimisation problems, they discuss them in detail and propose mathematical programming approaches. Subsequently, Guajardo and Ronnqvist (2015) analyse an inventory pool of spare parts, subject to a service level constraint, where the members of the pool may have different target service levels, so that they represent different demand classes. They show the important effects that different targets can have in the core stability for this cooperative model. Not long ago, Ciardiello et al. (2018) study the problem of pollution responsibility allocation across multi-tier supply chains. The model is further developed with reference to the case of a linear supply chain and several allocation rules are derived. In order to characterize such rules, desirable properties in terms of fairness, efficiency and transparency are introduced. Furthermore, a stability concept for efficient allocations is formulated.

Nevertheless still a better knowledge is needed in some models. In this paper we elaborate on the study of cooperation in production-inventoy settings that was introduced in Guardiola et al. (2009). There, the authors proposed a mode of relationship among heterogeneous agents that enforces a horizontal cooperation in this framework. This behaviour is captured in a cooperative game named Production-Inventory game (henceforth PI-game). That paper proposed the so-called *Owen point* core-allocation that allows all players to operate at minimum cost at the price of not compensating essential users by the cost reduction that they induce over the remaining players (followers). This allocation has proven to be rather appealing and in another paper, Guardiola et al. (2008) analyze its properties and provide axiomatic characterizations for the Owen point.

Both of those papers also contribute to a better knowledge of the core of PI-games. Nevertheless, it was missing a deeper analysis of its complexity. Specifically speaking the two following aspects were not considered: testing core membership and the extreme points structure of the core of these games. Complexity issues in cooperative game theory raise important questions only partially answered for particular classes of games. The core of any convex game is the convex hull of its marginal vectors (Shapley 1971), and the same property holds true for those games satisfying the Co-Ma property which include, among others, assignment and information games, see Hamers et al. (2002) and Kuipers (1993) respectively. It is also well-known that the core of assignment games coincide with the allocations induced by dual solutions and it is a complete lattice with only two extreme points, see Sotomayor (2003). Also, for transportation games, which constitute an extension of the assignment games, some results about the relationship between the core and the allocations induced by dual solutions are provided by Sánchez-Soriano et al. (2001). Moreover, Perea et al. (2012) study cooperation situations in linear production problems. In particular, that paper proposes a new solution concept called EOwen set as an improvement of the Owen set that contains at least one allocation that assigns a strictly positive payoff to players necessary for optimal production plans.

For minimum cost spanning tree games, flow games, linear production games, cooperative facility location games or min-coloring games among others, testing whether a given allocation is in the core is an NP-complete problem (see Faigle et al. 1997; Fang et al. 2002; Goemans and Skutella 2004 and Deng et al. 1999, respectively). On the other hand, there are some classes of games for which testing core membership is polynomially solvable as for instance for routing games, see Derks and Kuipers (1997), $s-t$ connectivity games, r-arborescence games, max matching games, min vertex cover games, min edge cover games or max independent set games, see e.g., Deng et al. (1999). However, for many other classes of cooperative games answering that question is still open, as it is the case of PI-games.

In this paper, we look for alternative cost allocations improving the fairness properties of the Owen point in that they recognize the role of the essential players on reducing the costs of the remaining players. In addition, we investigate the structure of the core of PI-games by determining its algorithmic complexity. Our contribution is to prove that testing core membership is an NP-complete problem and moreover that the number of extreme points of the core of PI-games is exponential on the number of players. Specifically, we characterize an exponential size subset of them.

To present our results the rest of the paper is organized as follows. We start by introducing some preliminary concepts in Sect. 2. In Sect. 3 we prove that testing core membership of PI-games in an NP-complete problem, and we analyze the core structure of PI-games. We define what we call the *extreme functions*, which help us to prove that the core of a PI-game, in general, has an exponential number of extreme points. In Sect. 4 we introduce a new core-allocation for PI-games, the Omega point, and provide an axiomatic characterization. Finally, in Sect. 5 we define the set of Quid Pro Quo allocations (henceforth, QPQ allocations). Every QPQ allocation is a convex combination of the Owen and the Omega point. We focus then on the equally weighted QPQ allocation, the *Solomonic* allocation, and we provide some necessary conditions for the coincidence of the latter with the Shapley value and the Nucleolus.

2 Preliminaries

A cost game with transferable utility (henceforth TU cost game) is a pair (N, c), where $N = \{1, 2, \ldots, n\}$ is the finite set of players, and the characteristic function $c : \mathcal{P}(N) \rightarrow \mathbb{R}$, is defined over $\mathcal{P}(N)$ the set of nonempty coalitions of N. By agreement, it always satisfies $c(\varnothing) = 0$. For all $S \subseteq N$, we denote by $|S|$ the cardinal of the set S.

A distribution of the costs of the grand coalition, usually called cost-sharing vector, is a vector $x \in \mathbb{R}^N$. For a given coalition $S \subseteq N$ we denote by $x_S := \sum_{i \in S} x_i$ the cost-sharing of coalition S (where $x_\varnothing = 0$). The core of a TU cost game is a set solution consisting of those cost-sharing vectors x which allocate the grand coalition cost $c(N)$ in such a way that no coalition S has incentives to leave N because $x(S)$ is smaller than the original cost of S, $c(S)$. Formally, the core of (N, c) is given by

$$Core(N, c) = \left\{x \in \mathbb{R}^n / x_N = c(N) \text{ and } x_S \leq c(S) \text{ for all } S \subset N\right\}.$$

In the following, core-allocations will be cost-sharing vectors belonging to the core. A cost game (N, c) is balanced if and only if has a nonempty core (see Bondareva 1963 or Shapley 1967). Shapley and Shubik (1969) describe totally balanced games as those games whose subgames are also balanced; i.e., the core of every subgame is nonempty. A cost game (N, c) is concave if for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subset N$ with $i \in S$, then $c(S) - c(S \backslash \{i\}) \geq c(T) - c(T \backslash \{i\})$.

It is well-known that the core is a bounded convex polyhedron, it has a finite number of extreme points. Moreover, the core is a convex set. Therefore, characterizing the extreme core-allocations is important to know the core structure. Let Q be a bounded convex polyhedron in $\mathbb{R}^n$. We say that $x \in Q$ is an extreme point if $y, z \in Q$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$. From now on, we denote, respectively, by $Ext\,(Q)$ and by $\partial\,(Q)$ the set of extreme points and the boundary of the set of Q. Moreover, for the sake of readability, we use e_i to refer to the i-th element of the canonical basis of $\mathbb{R}^n$ and $val(P)$ stands for the optimal value of the mathematical programming problem P. We will also use a characterization of the extreme points based on the restrictions that define the polyhedron Q. That is, $x \in Ext\,(Q)$ if and only if x satisfies as equalities at least n linearly independent constraints of those defining Q.

The Shapley value (Shapley 1953) is a function-point solution on the class of all TU games and for a cost game (N, c) it is defined as $\phi(N, c) = (\phi_i(N, c))_{i \in N}$ where for all $i \in N$

$$\phi_i(N, c) = \sum_{S \subseteq N \backslash \{i\}} \frac{s!(n - s - 1)!}{n!} \cdot [c(S \cup \{i\}) - c(S)].$$

The Nucleolus $\eta(N, c)$ (Schmeidler 1969) is the allocation that lexicographically minimizes the vector of excesses. It is well-known that the Nucleolus is a core-allocation provided that the core is nonempty.

From now on, and for the sake of readability, we follow the same notation as Guardiola et al. (2009) to describe Production-Inventory situations (henceforth: PI-situations) and PI-games. Consider first a situation with several agents facing each one a Production-Inventory problem. Then, they decide sharing technologies (production, inventory carrying and backlogged demand) to reduce costs. We mean that if a group of agents cooperates then they will produce and pay inventory carrying and backlogged demand at the cheapest costs among the members of the coalition at each period. This situation is called a PI-situation.

Formally, A PI-situation is a 3-tuple (N, D, Z) where $N = \{1, \ldots, n\}$ is a finite player set and D an integer matrix of demands with $D = [d^1, \ldots, d^n]'$, $d^i = [d_1^i, \ldots, d_T^i] \geq$

0, d_t^i is the demand of the player i in period $t \in T$ and T is the planning horizon. In addition, $Z = (H|B|P)$ is a cost matrix, so that $H = [h^1, \ldots, h^n]'$, $B = [b^1, \ldots, b^n]'$ and $P = [p^1, \ldots, p^n]'$; where $h^i = [h_1^i, \ldots, h_T^i] \geq 0$, are the inventory carrying costs, $b^i = [b_1^i, \ldots, b_T^i] \geq 0$, are the unit backlogging carrying costs and $p^i = [p_1^i, \ldots, p_T^i] \geq 0$, the production costs. The decision integer variables of the model, are for each period t: q_t, production during period, I_t, inventory at hand at the end of period and E_t, the backlogged demand at the end of period. The set of PI-situations (N, D, Z) is denoted by Υ, with $n \geq 1$, $T \geq 1$ and D an integer matrix.

Now given a PI-situation (N, D, Z), we can associate the corresponding TU cost game (N, c) with the following characteristic function c: $c(\emptyset) = 0$ and for any $S \subseteq N$, $c(S) = val(PI(S))$, where $PI(S)$ is given by

$$\begin{aligned}
&(PI(S)) \quad \min \sum_{t=1}^{T} (p_t^S q_t + h_t^S I_t + b_t^S E_t) \\
&\text{s.t.} E_0 = E_T = I_0 = I_T = 0, \\
&E_t - I_t = E_{t-1} - I_{t-1} + d_t^S - q_t, \quad t = 1, \ldots, T, \\
&q_t,\ I_t,\ E_t,\ \text{non-negative, integer}, t = 1, \ldots, T;
\end{aligned}$$

with

$$p_t^S = \min_{i \in S}\{p_t^i\},\ h_t^S = \min_{i \in S}\{h_t^i\},\ b_t^S = \min_{i \in S}\{b_t^i\},\ d_t^S = \sum_{i \in S} d_t^i.$$

Every TU cost game defined as above is called a Production-Inventory game. Guardiola et al. (2009) points out that the problem $PI(S)$ has integer optimal solutions provided that the demands are integer. We know that for any $S \subseteq N$ the dual problem of $PI(S)$ is the following mathematical programming problem,

$$\begin{aligned}
&(DLPI(S)) \quad \max \sum_{t=1}^{T} d_t^S y_t \\
&\text{s.t.} y_t \leq p_t^S, \qquad t = 1, \ldots, T, \\
&y_{t+1} - y_t \leq h_t^S, \quad t = 1, \ldots, T-1, \\
&-y_{t+1} + y_t \leq b_t^S, \quad t = 1, \ldots, T-1.
\end{aligned}$$

Moreover, Guardiola et al. (2009) also proves that an optimal solution of problem $DLPI(S)$ is $y_t^*(S) = \min\left\{p_t^S, \min_{k<t}\{p_k^S + h_{kt}^S\}, \min_{k>t}\{p_k^S + b_{tk}^S\}\right\}$, for all $t = 1, \ldots, T$, with

$$\begin{aligned}
p_k^S &= \begin{cases} p_1^S & \text{if } k < 1, \\ p_T^S & \text{if } k > T, \end{cases} \\
h_{kt}^S &= \sum_{r=k}^{t-1} h_r^S, \quad \text{for any } k < t, t = 2, \ldots, T;\ h_{k1}^S = 0, k < 1, \\
b_{tk}^S &= \sum_{r=t}^{k-1} b_r^S, \quad \text{for any } k > t,\ t = 1, \ldots, T-1;\ b_{Tk}^S = 0, k > T.
\end{aligned}$$

It is important to note that those optimal solutions satisfy a monotonicity property with respect to coalitions : $y_t^*(S) \geq y_t^*(R)$ for all $S \subseteq R \subseteq N$ and all $t \in \{1, \ldots, T\}$. Moreover, $c(S) = \sum_{t=1}^{T} y_t^*(S) \cdot d_t^S$ for any $\emptyset \neq S \subseteq N$.

PI-games are not concave in general. The Owen point, denoted by $Owen(N, D, Z)$, is the allocation $\left(\sum_{t=1}^{T} d_t^i y_t^*(N)\right)_{i \in N}$. At times, for the sake of simplicity, we use o for the Owen point. Guardiola et al. (2009) also proves that the Owen point belongs to the core which can be reached through a PMAS (Sprumont 1990); hence PI-games are totally balanced. When we need to point out that the game (N, c) comes from the situation (N, D, Z) we will use $c^{(N,D,Z)}(S)$.

If there exists a period $t \in \{1, \ldots, T\}$ such that $y_t^*(N\backslash\{i\}) > y_t^*(N)$ with $d_t^{N\backslash\{i\}} > 0$ we will call i essential player. An essential player is the one that is needed by the rest of players (at least one period) to produce a certain demand at a minimum cost. The set of essential players is denoted by $\mathcal{E}$. Those players not being essential are called inessential. We can easily check that if i is an inessential player, $o_{N\backslash\{i\}} = c(N\backslash\{i\})$.

Finally, to conclude this section devoted to preliminaries, we recall the class of PS-games introduced by Kar et al. (2009). A PS-game (N, c) is a TU cost game satisfying that for all player $i \in N$, there exists a real constant c_i such that $\Delta_i(S) + \Delta_i(N\backslash(S \cup \{i\}) = c_i$ for all $S \subseteq N\backslash\{i\}$, where $\Delta_i(S) := c(S \cup \{i\}) - c(S)$. The above mentioned paper proves that, for this class of games, the Shapley value and the Nucleolus coincide; i.e. $\phi(N, c) = \eta(N, c)$.

3 Extreme points of the core of PI-games

Guardiola et al. (2009) demostrated that the core of PI-games without essential players ($\mathcal{E} = \varnothing$) shrinks to a singleton, the Owen point. However, for those PI-games with essential players ($\mathcal{E} \neq \varnothing$), the core is large. We focus here on those PI-games with large cores and study the structure of its core by analyzing its extreme points. First of all, we remark that testing core membership for PI-games cannot be done in polynomial time. One can adapt the reduction proposed in Fang et al. (2002) to prove that checking if an imputation belongs to the core of a PI-game is an NP-complete decision problem. In spite of that, it is important to know the structure of the core and still very little is known about the extreme points complexity of PI-games. This is the goal of this section.

We begin this analysis by defining the essential player follower set.

Let (N, D, Z) be a PI-situation with D being an integer matrix ($(N, D, Z) \in \Upsilon$), and let i be an essential player. We define the follower set of i as follows:

$$F_i := \{j \in N\backslash\{i\} \,\Big|\, \exists t \in \{1, \ldots, T\} \text{ with } d_t^j > 0 \text{ and } y_t^*(N\backslash\{i\}) > y_t^*(N)\}.$$

The follower set of player i consists of all players who need him to operate at a lower cost. It is always a non-empty set. Indeed, $F_i \neq \varnothing$ since taking $i \in \mathcal{E}$, there exists $t^* \in \{1, \ldots, T\}$ such that $y_{t^*}^*(N\backslash\{i\}) > y_{t^*}^*(N)$ and $d_{t^*}^{N\backslash\{i\}} > 0$. In that case, there must be, at least, a player $j \in N\backslash\{i\}$ such that $d_{t^*}^j > 0$ and $y_{t^*}^*(N\backslash\{i\}) > y_{t^*}^*(N)$.

In addition, you may notice that there is a pairwise relationship among essential players and their followers, in the sense that the latter are interested in taking on a portion of the costs of the former. This relationship allows us to introduce the concept of essential-follower pair.

Let $(N, D, Z) \in \Upsilon$. The essential-follower pair set, denoted by $\mathbb{P}$, is:

$$\mathbb{P} := \{(i, j) | i \in \mathcal{E} \text{ and } j \in F_i\}.$$

We are now interested in determining the cost that can be transferred within every essential-follower pair with a cost allocation; i.e., the maximum portion of the essential player cost that his follower could assume while maintaining cooperation.

Given an essential-follower pair $p = (i, j) \in \mathbb{P}$ and an allocation $x \in \mathbb{R}^n$, the transferred cost induced by p regarding x is:

$$\alpha_p(x) := \min_{R\in\Delta_p} \{c(R) - x_R\},$$

where

$$\Delta_{(i,j)} := \{R \subseteq N\backslash\{i\} \text{ such that } j \in R\}.$$

$\alpha_p(x)$ can be interpreted as the maximum portion of cost of player i that can be awarded by player j while maintaining the cooperation of the group. It is worth nothing that if $x \in Core(N, c)$ then $\alpha_p(x) \geq 0$.

Next result states that there are always a positive transferred cost within every essential-follower pair with the Owen point.

Lemma 3.1 *Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. Then $\alpha_p(o) > 0$ for all $p \in \mathbb{P}$.*

Proof As $\mathcal{E} \neq \varnothing$, we can take $i \in \mathcal{E}$ and therefore $F_i \neq \varnothing$. Let $R \subseteq N\backslash\{i\}$ such that $R \cap F_i \neq \varnothing$ and let $j \in R \cap F_i$. By definition, there exists $t^* \in \{1, \ldots, T\}$ such that $y^*_{t^*}(N\backslash\{i\}) > y^*_{t^*}(N)$ and $d^j_{t^*} > 0$. Then $d^R_{t^*} > 0$ and moreover $y^*_{t^*}(N) < y^*_{t^*}(N\backslash\{i\}) \leq y^*_{t^*}(R)$. Thus, $o_R < c(R)$. Hence, $\alpha_p(o) = \min_{R\in\Delta_p}\{c(R) - o_R\} > 0$. □

We introduce now a function that transforms any cost allocation into a new cost allocation in which a follower player charges with the maximum cost of his essential player. That is, for each $p = (i, j) \in \mathbb{P}$, the function f_p transforms any allocation x into a new allocation $f_p(x)$, in which the follower player j assumes as much cost as possible from his essential player i. It is called the extreme function.

Definition 3.2 (*extreme function*) Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. For any $p = (i, j) \in \mathbb{P}$, the extreme function f_p is defined by:

$$f_p(x) = x + \wedge_p(x),$$

where $x \in \mathbb{R}^n$ and $\wedge_p(x) = e_j \cdot \alpha_p(x) - e_i \cdot \alpha_p(x)$.

Let us denote by $\mathbb{P}^{|\mathbb{P}|}$ the $|\mathbb{P}|$-fold cartesian product of the set $\mathbb{P}$. We consider now the composition of extreme functions. For each $\sigma \in \mathbb{P}^{|\mathbb{P}|}$ we define the extreme composite function, F_σ, as the composition of extreme functions for all the pairs in σ, that is,

$$F_\sigma(x) := \left(f_{\sigma_{|\mathbb{P}|}} \circ f_{\sigma_{|\mathbb{P}|-1}} \circ \cdots \circ f_{\sigma_1}\right)(x).$$

Notice that if $\sigma = (p, p, \ldots, p) \in \mathbb{P}^{|\mathbb{P}|}$ then $F_\sigma(x) = f_p(x)$.

Example 3.3 The following table shows a PI-situation with three players and three periods:

We can easily check that $c(S) = \sum_{t=1}^{3} p^S_t d^S_t$, for all $S \subseteq N$. Hence, the characteristic function of the corresponding PI-game is given in the following table:

In this example, $y^*(N) = (1, 1, 1)$ and $y^*(N\backslash\{1\}) = (2, 1, 1)$ are the optimal solution for $(DLPB(N))$ and $(DLPB(N\backslash\{1\}))$, respectively. Then, the Owen point is $o = (25, 26, 13)$. Moreover, $\mathcal{E} = \{1\}$, $F_1 = \{2, 3\}$ and $\mathbb{P} = \{(1, 2), (1, 3)\}$.

The transferred cost within every essential-follower pair in $\mathbb{P}$ with the Owen point are,

$$\alpha_{(1,2)}(o) = \min_{R\in\Delta_{(1,2)}} \{c(R) - o_R\} = 10,$$

$$\alpha_{(1,3)}(o) = \min_{R\in\Delta_{(1,3)}} \{c(R) - o_R\} = 12.$$

Therefore, the extreme functions are

$$f_{(1,2)}(o) = o + \wedge_{(1,2)}(o) = (15, 36, 13),$$
$$f_{(1,3)}(o) = o + \wedge_{(1,3)}(o) = (13, 26, 25).$$

In this case, both the extreme functions, $f_{(1,2)}(o)$, $f_{(1,3)}(o)$, and the Owen point, o, are extreme points of the core.

	d_1^S	d_2^S	d_3^S	p_1^S	p_2^S	p_3^S	h_1^S	h_2^S	b_1^S	b_2^S	c
{1}	10	10	5	1	2	1	1	1	1	1	35
{2}	8	12	6	2	1	1	1	1	1	1	36
{3}	6	5	2	3	1	1	1	1	2	2	25
{1, 2}	18	22	11	1	1	1	1	1	1	1	51
{1, 3}	16	15	7	1	1	1	1	1	1	1	38
{2, 3}	14	17	8	2	1	1	1	1	1	1	53
{1, 2, 3}	24	27	13	1	1	1	1	1	1	1	64

	Demand			Production			Inventory		Backlogging	
P1	10	10	5	1	2	1	1	1	1	1
P2	8	12	6	2	1	1	1	1	1	1
P3	6	5	2	3	1	1	1	1	2	2

The previous example shows that the Owen point is an extreme point of the core, and that the extreme functions transform it into other extreme points of the core. We wonder then if this fact occurs in general for any PI-game. First, we find a very interesting property that relates the extreme functions to the core boundary.

Proposition 3.4 *Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. For all $p \in \mathbb{P}$*

$$f_p(Core(N, c)) \subseteq \partial(Core(N, c)).$$

Proof Let $p = (i, j) \in \mathbb{P}$ and take $x \in Core(N, c)$. Then $\alpha_p(x) \geq 0$. Applying the extreme function f_p at x, we have:

$$f_p(x) = (x_1, \dots, x_{i-1}, x_i - \alpha_p(x), x_{i+1}, \dots, x_{j-1}, x_j + \alpha_p(x), x_{j+1}, \dots, x_n).$$

To prove that $y := f_p(x) \in Core(N, c)$ we distinguish four possibilities:

- $i, j \in S$. Then $y_S = x_S + \alpha_p(x) - \alpha_p(x) = x_S \leq c(S)$.
- $i, j \notin S$. Then $y_S = x_S \leq c(S)$.
- $i \notin S, j \in S$. Then $y_S = x_S + \alpha_p(x) \leq x_S + c(S) - x_S = c(S)$.
- $i \in S, j \notin S$. Then $y_S = x_S - \alpha_p(x) \leq x_S \leq c(S)$.

Hence, $y \in Core(N, c)$ since $y_S \leq c(S)$ for any coalition $S \subseteq N$. Let us proof now that y belongs to the frontier of the core.

If $\alpha_p(x) = 0$ then there exists $R \in \Delta_p$ such that $c(R) = x_R$. Since y belongs to the core and satisfies as equality one of the constraints defining the core, we can conclude that $y \in \partial(Core(N, c))$.

If $\alpha_p(x) > 0$ then for all $\lambda \in (0, 1)$, $(1-\lambda)x + \lambda y \in Core(N, c)$. Take $\lambda = 1 + \epsilon$ with $\epsilon > 0$ to have

$$(1-\lambda)x + \lambda y = -\epsilon x + (1+\epsilon)(x + \wedge_p(x)) = x + (1+\epsilon) \wedge_p (x).$$

We can check that if $R^* \in \Delta_p$ is such that $c(R^*) - x_{R^*} = \min_{R \in \Delta_p}\{c(R) - x_R\}$ then $x_{R^*} + \alpha_p(x) = c(R^*)$, therefore $x + (1+\epsilon) \wedge_p (x) \notin Core(N, c)$. Hence, y is not an interior point.

It follows straightforward from the above proposition, that $F_\sigma(Core(N, c)) \subseteq \partial(Core\,(N, c))$ for all $\sigma \in \mathbb{P}^{|\mathbb{P}|}$.

The main Theorem of this Section provides a partial answer to our previous question about the transformation of the Owen point into extreme points of the core of PI-games. It states that for PI-situations with a single essential player, all the different compositions of extreme functions over the Owen point generate extreme points of the core.

Theorem 3.5 *Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. If $\mathcal{E} = \{i\}$, then $F_\sigma(o) \in Ext\,(Core(N, c))$ for all $\sigma \in \mathbb{P}^{|\mathbb{P}|}$.*

Proof Let $j \in F_i$ then the pair $p_j = (i, j) \in \mathbb{P}$. $f_{p_j}(o)$ is an extreme point if for any $y, z \in Core\,(N, c)$ such that

$$f_{p_j}(o) = \frac{1}{2}y + \frac{1}{2}z \text{ we have that } y = z. \tag{1}$$

By definition, we know that

$$f_{p_j}(o) = (o_1, \ldots, o_{i-1}, o_i - \alpha_{p_j}(o), o_{i+1}, \ldots, o_{j-1}, o_j + \alpha_{p_j}(o), o_{j+1}, \ldots, o_n).$$

Let us suppose that $z_k < o_k$ for any $k \neq i, j$ then $z_{N\backslash\{k\}} > o_{N\backslash\{k\}} = c\,(N\backslash\{k\})$. However this is not possible, therefore $y_k, z_k \geq o_k$ for all $k \neq i, j$. Now, apply (1) to get that $y_k = z_k = o_k$ $\forall k \neq i, j$. Moreover, $y_j, z_j \leq o_j + \alpha_{p_j}(o)$ since $\alpha_{p_j}(o) > 0$ is the maximum possible increment for o_j (see Lemma 3.1). Then by (1) we have that $z = y$ and hence $f_{p_j}(o) \in Ext\,(Core(N, c))$.

Now, we consider $p_l = (i, l) \in \mathbb{P}$, and apply the corresponding extreme function for this pair. We have that

$$f_{p_l}\left(f_{p_j}(o)\right) = (o_1, \ldots, o_i - \alpha_{p_j}(o) - \alpha_{p_l}(f_{p_j}(o)), \ldots, o_l + \alpha_{p_l}(f_{p_j}(o)), \ldots, o_j + \alpha_{p_j}(o), \ldots, o_n).$$

We distinguish two possibilities:

1. $\alpha_{p_j}(o)$ attains its minimum in a coalition R^* that contains player l. In this case $\alpha_{p_l}(f_{p_j}(o)) = 0$, thus $f_{p_l}\left(f_{p_j}(o)\right) = f_{p_j}(o)$ and by the argument above $f_{p_l}\left(f_{p_j}(o)\right)$ is an extreme point of $Core(N, c)$.
2. $\alpha_{p_j}(o)$ attains its minimum in a coalition R^* that does not contain player l. This case implies that $\alpha_{p_l}(f_{p_j}(o)) > 0$. Take $y, z \in Core(N, c)$ and assume that

$$f_{p_l}\left(f_{p_j}(o)\right) = \frac{1}{2}y + \frac{1}{2}z. \tag{2}$$

Using the same argument as above we conclude that $y_k = z_k = o_k$ for all $k \neq i, j, l$. Consider now the j-th coordinate. Suppose that $z_j > o_j + \alpha_{p_j}(o)$. The coalition R^* does not contain neither i nor l, which implies $c(R^*) = o_{R^*} + \alpha_{p_j}(o) < z_{R^*}$. Since this is a contradiction, it means that $z_j \leq o_j + \alpha_{p_j}(o)$ (Notice that the same argument applies to y_j and thus $y_j \leq o_j + \alpha_{p_j}(o)$). Therefore, by (2) we get that $y_j = z_j = o_j + \alpha_{p_j}(o)$. Next, consider the l-th coordinate. Assume that $z_l > o_l + \alpha_{p_l}(f_{p_j}(o))$, and let S^* be the coalition where $\alpha_{p_l}(f_{p_j}(o))$ attains its minimum, then $c(S) = \sum_{k\in S}\left(f_{p_j}(o)\right)_k + \alpha_{p_l}(f_{p_j}(o)) < z_S$. Again using the same argument as in the j-th coordinate we conclude that $y_l = z_l = o_l + \alpha_{p_l}(f_{p_j}(o))$. Finally, we get the same conclusion for the i-th coordinate since $f_{p_l}\left(f_{p_j}(o)\right)$ must be efficient. In conclusion $z = y$. Hence, $f_{p_l}\left(f_{p_j}(o)\right) \in Ext\,(Core(N,c))$. Notice that $f_{p_l}\left(f_{p_j}(o)\right)$ is different from $f_{p_j}(o)$ since we have assumed that $\alpha_{p_l}(f_{p_j}(o)) > 0$.

This construction can be repeated a finite number of times for each $p \in \mathbb{P}$. Specifically, for any $\sigma \in \mathbb{P}^{|\mathbb{P}|}$, the transformation $F_\sigma(o) \in Ext\,(Core(N,c))$. □

Corollary 3.6 *Let $(N, D, Z) \in \Upsilon$ with $\mathcal{E} = \{i\}$, and (N, c) be the corresponding PI-game. The Owen point is always an extreme point.*

Proof Take $y_k, z_k \geq o_k$ for all $k \in N$, therefore $o = z = y$ and $o \in Ext\,(Core(N,c))$. □

At this point we know that PI-games with a single essential player have, at least, $|\mathbb{P}| + 1$ extreme points. Next example shows that the core of a PI-game, in general, cannot be explicitly described in polynomial time.

Example 3.7 Now we consider a PI-situation with n periods and n players (Table 1):

The corresponding PI-game is given by $c(S) = \sum_{t=1}^{n} p_t^S d_t^S$, for all $S \subseteq N$. Moreover, it is easy to see that $\mathcal{E} = \{1\}$ and $F_1 = \{2, 3, \ldots, n\}$. Then, we can rewrite the characteristic function as follows:

$$c(S) = \begin{cases} |S| & \text{if } 1 \in S, \\ \\ n \cdot |S| & \text{if } 1 \notin S. \end{cases}$$

In this example, the Owen point is $o = (1, 1, \ldots, 1)$. For all $i \in F_1$,

$$\alpha_{(1,i)}(o) = \min_{R\in\Delta_{(1,i)}} \{c(R) - o_R\} = \min_{R\in\Delta_{(1,i)}} \{n \cdot |R| - |R|\} = n - 1$$

then

$$f_{(1,i)}(o) = (2 - n, 1, \ldots, 1, \underbrace{n}_{i}, 1, \ldots, 1).$$

For all $k \neq i$, $k \in F_1$,

$$\alpha_{(1,k)}(f_{(1,i)}(o)) = \min_{R\in\Delta_{(1,k)}} \{c(R) - \sum_{j\in R} \left(f_{(1,i)}(o)\right)_j\}$$

Table 1 Game of the Example 3.7

	Demand				Production				Inventory				Backlogging			
P1	1	1	…	1	$\frac{1}{n}$	$\frac{1}{n}$	…	$\frac{1}{n}$	2	2	…	2	2	2	…	2
P2	1	1	…	1	1	1	…	1	2	2	…	2	2	2	…	2
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮
Pn	1	1	…	1	1	1	…	1	2	2	…	2	2	2	…	2

$$= \min_{R \in \Delta_{(1,k)}} \{\underbrace{n \cdot |R| - |R|}_{\text{if } i \notin R}, \underbrace{(n-1) \cdot |R| - n + 1}_{\text{if } i \in R}\} = n - 1,$$

then

$$f_{(1,k)}\left(f_{(1,i)}(o)\right) = (3 - 2n, 1, \ldots, 1, \underbrace{n}_{i}, 1, \ldots, 1, \underbrace{n}_{k}, 1, \ldots, 1).$$

Hence, we have as many extreme points as possible ways to place "n'' and "$1''$ in $n-1$ positions; i.e. in this example the core has $2^{n-1}+1$ extreme points.

Therefore, we can conclude that the cardinality of the extreme points is exponential in the number of players. Hence, we cannot explicitly describe the core of a PI-game in polynomial time.

We propose below an alternative core allocation to the Owen point that recognizes the role played by essential players on reducing the cost of their followers.

4 Omega point

Guardiola et al. (2009) proposed the Owen point as a natural core allocation for PI-games that arises when focusing on shadow prices of each period that each player must pay to meet their demand in that period. It makes it possible for all players in the joint venture to operate at minimum cost. If there is no essential player, the Owen point is the unique core allocation. However, for those PI-situations with at least one essential player, the Owen point reveals the altruistic character of them because of it does not take into account the role that these essential players play in reducing the cost of their followers. As the core of the PI-games with essential players is large, we are looking for a core allocation that motivates the essential players to continue in the join venture obtaining a reduction in their demand costs in each period.

Let (N, D, Z) be a PI-situation with D being an integer matrix ($(N, D, Z) \in \Upsilon$), and $\mathcal{E} \neq \varnothing$. Remember that for all $i \in \mathcal{E}$, there is a period $t^* \in \{1, \ldots, T\}$ such that $y^*_{t^*}(N\backslash\{i\}) > y^*_{t^*}(N)$ and there also exists at least one player $j \in N\backslash\{i\}$ such that $d^j_{t^*} > 0$. We denote by $\mathcal{E}^t$ and F^t the sets of essential players and followers for every period $t \in \{1, \ldots, T\}$. We note in passing that $\mathcal{E} = \bigcup_{t \in T} \mathcal{E}^t$.

First, we consider the marginal contribution of the shadow prices of a player i to the grand coalition N, that is, $y^*_t(N\backslash\{i\}) - y^*_t(N)$. We then define the cost reduction that a player $i \in N$ can produce in another player $j \in N$ in a period t as follows:

$$q_t(i, j) := \begin{cases} \left(y^*_t(N\backslash\{i\}) - y^*_t(N)\right) \cdot d^j_t & if\ i \neq j \\ 0 & if\ i = j \end{cases}$$

The reader may notice that $q_t(i, j) > 0$ only if $i \in \mathcal{E}^t$ and $j \in F^t$, otherwise $q_t(i, j) = 0$. That is to say that only essential players can reduce their follower costs in a given period. Alternatively, the amount of the cost $q_t(i, j)$ can be interpreted as the maximum cost increase that a follower $j \in F^t$ is able to assume, in a certain period t, to incentivize the essential player $i \in \mathcal{E}^t$.

Next we define a new cost allocation rule, the Omega point, that considers the maximum cost increase mentioned above.

Definition 4.1 (Omega point) Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. The Omega point $\omega \in \mathbb{R}^n$ is defined as $\omega_i = \sum_{t=1}^{T} \omega^t_i$ for all player $i \in N$, where for each period $t = 1, \ldots, T$,

$$\omega_i^t := \begin{cases} y_t^*(N)d_t^i + Q_i^t & \text{if } \left|\mathcal{E}^t\right| = 1 \\ y_t^*(N)d_t^i & \text{otherwise} \end{cases}$$

with $Q_i^t = q_t(\mathcal{E}^t, i) - \sum_{j \in F^t} q_t(i, j)$.

The Omega point means that, in each of the periods with a single essential player, i.e. without competition, this essential player gets a cost reduction from his followers. The amount Q_i^t represents the cost reduction or increase, depending on the sign, for player $i \in N$ in the period t. Notice that $Q_i^t < 0$, only if i is an essential player, otherwise $Q_i^t \geq 0$. In addition, $Q^t = \left(Q_i^t\right)_{i \in N}$ for all $t \in \{1, \ldots, T\}$.

The reader may also note that $\omega = o + Q$, where $Q \in \mathbb{R}^n$ with $Q_i = \sum_{t=1}^{T} Q_i^t$. It is worth noting that Q_i represents the marginal cost reduction or increase, of player i to the rest players. Moreover, $\sum_{i \in N} Q_i = 0$. In this setting, those players with $Q_i < 0$, would prefer the Omega point to the Owen point. On the contrary, those players with $Q_i > 0$ would like the Owen point more.

The following example illustrates the cost reduction that the Omega point applies to essential players while increasing the cost of followers.

Example 4.2 In example 3.3 the Owen point is $o = (25, 26, 13)$. Moreover, $\mathcal{E}^1 = \{1\}$, $F^1 = \{2, 3\}$, $\mathcal{E}^2 = \mathcal{E}^3 = \varnothing$. The cost reduction for the essential player 1 from his followers 2 and 3 are:

$$q_1(1, 2) = 8; q_1(1, 3) = 6;$$

Therefore,

$$\begin{aligned} \omega_1 &= o_1 - q_1(1, 2) - q_1(1, 3) = 25 - 8 - 6 = 11 \\ \omega_2 &= o_2 + q_1(1, 2) = 34 \\ \omega_3 &= o_3 + q_1(1, 3) = 19 \end{aligned}$$

In this case $\omega = (11, 34, 19) = (25, 26, 13) + Q$, with $Q = (-14, 8, 6)$. It is also a core-allocation. Note that player 1 obtains a cost reduction of 14 units, while players 2 and 3 are increasing their costs by 8 and 6 units, respectively. Here, the Omega point is a core-allocation that recognizes the essential role of player 1 through a cost reduction assumed by his followers. Next we demonstrate that this always holds for any PI-game.

Proposition 4.3 *Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. The Omega point is always a core-allocation.*

Proof Consider any period t and a coalition $S \subseteq N$. If t does not have essential players or has more than one, then $\omega_S^t = \sum_{i \in S} y_t^*(N)d_t^i \leq \sum_{i \in S} y_t^*(S)d_t^i = y_t^*(S)d_t^S$.

Otherwise, suppose that player k is essential in the period t ($\mathcal{E}^t = \{k\}$). we distinguish two possibilities:

- $k \in S$, then

$$\begin{aligned} \omega_S^t = \omega_k^t + \sum_{i \in S \cap F^t} \omega_i^t + \sum_{i \in S \setminus F^t} \omega_i^t &= y_t^*(N)d_t^k - \sum_{j \in F^t} q_t(k, j) \\ &+ \sum_{i \in S \cap F^t} \left(y_t^*(N)d_t^i + q_t(k, i)\right) + \sum_{i \in S \setminus F^t} y_t^*(N)d_t^i \\ &= y_t^*(N)d_t^S + \sum_{i \in S \cap F^t} q_t(k, i) - \sum_{j \in F^t} q_t(k, j) \end{aligned}$$

$$= y_t^*(N)d_t^S - \sum_{j\in F^t\setminus S} q_t(k,j) \le y_t^*(N)d_t^S \le y_t^*(S)d_t^S$$

- $k \notin S$, then

$$\begin{aligned}
\omega_S^t &= \sum_{i\in S\cap F^t} \omega_i^t + \sum_{i\in S\setminus F^t} \omega_i^t = \sum_{i\in S\cap F^t} \left(y_t^*(N)d_t^i + q_t(k,i)\right) + \sum_{i\in S\setminus F^t} y_t^*(N)d_t^i \\
&= y_t^*(N)d_t^S + \sum_{i\in S\cap F^t} \left(y_t^*(N\setminus\{k\}) - y_t^*(N)\right) d_t^i \\
&= \sum_{i\in S\setminus F^t} y_t^*(N)d_t^i + \sum_{i\in S\cap F^t} y_t^*(N\setminus\{k\})d_t^i \\
&\le \sum_{i\in S} y_t^*(N\setminus\{k\})d_t^i \le \sum_{i\in S} y_t^*(S)d_t^i = y_t^*(S)d_t^S
\end{aligned}$$

Hence, $\omega_S^t \le y_t^*(S)d_t^S$ for all $t \in T$. Then, $\omega_S = \sum_{t=1}^T \sum_{i\in S} \omega_i^t = \sum_{t=1}^T \omega_S^t \le \sum_{t=1}^T y_t^*(S)d_t^S = c(S)$ for any coalition $S \subseteq N$. Moreover, $\omega_N = o_N + \sum_{i\in N} Q_i = o_N = c(N)$. Therefore $\omega \in Core(N,c)$. □

4.1 Characterization of the Omega point

To complete the study of the Omega point, we here propose an axiomatic characterization based on a set of desirable properties that make it unique. In order to do that, we denote by γ a generic allocation rule on Υ and consider the following properties, some of which have been used in the literature to axiomatize alternative allocations:

(EF) *Efficiency*. For all $x \in \gamma(N,D,Z)$ and for any PI-situation $(N,D,Z) \in \Upsilon$, $x_N = c^{(N,D,Z)}(N)$.

(NE) *Nonemptiness*. For any PI-situation $(N,D,Z) \in \Upsilon$, $\gamma(N,D,Z) \neq \varnothing$.

(IBC) *Inessential bounded cost*. For any PI-situation $(N,D,Z) \in \Upsilon$ and for all $x \in \gamma(N,D,Z)$, if i is an inessential player, then $x_i \le \sum_{t=1}^T y_t^*(N\setminus\mathcal{E}^t)d_t^i$

(TI) *Tyranny*. For all $x \in \gamma(N,D,Z)$ and for any PI-situation $(N,D,Z) \in \Upsilon$, if k is a single essential player then $x_{N\setminus\{k\}} = c^{(N,D,Z)}(N\setminus\{k\})$.

(ACP) *Additive combination of periods' demands*. For all $x \in \gamma(N,D,Z)$ and for any PI-situation $(N,D,Z) \in \Upsilon$, there exists $(z_t)_{t\in T} \in (\mathbb{R}^N)^N$ such that $x = \sum_{t=1}^T z_t$ and for all $t \in T$, $z_t \in \gamma(N,D_t,Z)$ if $\left|\mathcal{E}^t\right| \le 1$ and $z_t = Owen(N,D_t,Z)$ otherwise, where

$$D_t = \left(d^{ip}\right)_{\substack{i=1,\dots,n\\ p=1,\dots,T}},\; d^{ip} = \begin{cases} d_p^i & \text{if } t=p,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

The first two properties were already used in Guardiola et al. (2008), among many other papers, to characterize the Owen point solution, and they are also important to our new characterization of the Omega point. Recall that *Efficiency* ensures that the total cost is entirely allocated among the players. Analogously, *Nonemptiness* guarantees that this allocation rule always return a feasible allocation of the overall cost when applied to any PI-situation. *Inessential bounded cost* imposes a maxim cost for every inessential player in situations which an essentials players has left. *Tyranny* implies that a single essential player will assert all his power over the rest so that they assume the maximum possible cost.

Finally, an allocation rule satisfies the property of *Additivity combination of periods' demands* if it is additive with respect to the demand of the periods that has at most an essential player plus the Owen point of those periods with more than one essential player. We emphasize that this additivity results from the following relationship $c^{(N,D,Z)} = \sum_{t=1}^{T} c^{(N,D_t,Z)}$ for all $(N, D, Z) \in \Upsilon$. Thus, we are interested on allocation rules compatible with this form of distribution of their demands.

First, we show that the Omega point satisfies all the previous properties.

Proposition 4.4 *The Omega point defined on the set Υ, satisfies EF, NE, IBC, TI and ACP.*

Proof For any PI situation $(N, D, Z) \in \Upsilon$ we know by proposition 4.3 that $\omega(N, D, Z) \in Core(N, c)$ by . Hence, the Omega point verifies the properties of EF and NE. An inessential player satisfy IBC since if for all $i \in N$

$$\begin{aligned}\omega_i(N, D, Z) &= \sum_{t=1}^{T} \omega_i^t(N, D, Z) = \sum_{t=1}^{T} \omega_i(N, D_t, Z)\\ &= \sum_{t\in T/|\mathcal{E}^t|=1} \left(y_t^*(N)d_t^i + q_t(\mathcal{E}^t, i)\right) + \sum_{t\in T/|\mathcal{E}^t|\neq 1} y_t^*(N)d_t^i\\ &= \sum_{t\in T/|\mathcal{E}^t|=1} y_t^*(N\backslash\mathcal{E}^t)d_t^i + \sum_{t\in T/|\mathcal{E}^t|\neq 1} y_t^*(N)d_t^i \le \sum_{t=1}^{T} y_t^*(N\backslash\mathcal{E}^t)d_t^i\end{aligned}$$

if there is only one essential player k, then:

$$\begin{aligned}\sum_{t=1}^{T} \omega_{N\backslash\{k\}}^t &= \sum_{t=1}^{T} \left(y_t^*(N)d_t^{N\backslash\{k\}} + \sum_{j\in N\backslash\{k\}} q_t(k, i) \right)\\ &= \sum_{t=1}^{T} \left(y_t^*(N)d_t^{N\backslash\{k\}} + \sum_{j\in N\backslash\{k\}} \left(y_t^*(N\backslash\{k\}) - y_t^*(N)\right) d_t^j \right)\\ &= \sum_{t=1}^{T} \left(y_t^*(N\backslash\{k\})d_t^{N\backslash\{k\}} \right) = c^{(N,D,Z)}(N\backslash\{k\}).\end{aligned}$$

Then satisfy TI. Finally, considering D_t as it was already defined in (3), we obtain that $\sum_{t=1}^{T} D_t = D$ and

$$\begin{aligned}\omega(N, D, Z) &= \left(\sum_{t=1}^{T} \omega_i^t(N, D, Z) \right)_{i\in N} = \sum_{t=1}^{T} (\omega_i(N, D_t, Z))_{i\in N}\\ &= \sum_{t\in T/|\mathcal{E}^t|\le 1} \omega_i(N, D_t, Z) + \sum_{t\in T/|\mathcal{E}^t|\ge 2} Owen(N, D_t, Z)\end{aligned}$$

Hence, the Omega point satisfies ACP.

Second, we focus on PI-situations without essential players and show that, in this setting, the Omega point matches the Owen point, and both can be characterized by using only three of the previous properties.

Proposition 4.5 *Let $(N, D, Z) \in \Upsilon$ be a PI situation with $|\mathcal{E}| = 0$. Then, $\gamma(N, D, Z) = \omega(N, D, Z) = Owen(N, D, Z)$ if and only if γ satisfies NE, EF and IBC.*

Proof (If) The *if* part of the proof is direct from Proposition 4.4.
(Only if) $\gamma(N, D, Z) \neq \varnothing$ by NE. Take $x \in \gamma(N, D, Z)$. Since there are not essential players, by IBC, it holds that $x_i \leq \sum_{t=1}^{T} y_t^*(N \backslash \mathcal{E}^t) d_t^i = \sum_{t=1}^{T} y_t^*(N) d_t^i = \omega_i(N, D, Z)$ for each $i \in N$. Therefore, by EF, $\gamma(N, D, Z) = \omega(N, D, Z) = Owen(N, D, Z)$. □

The main Theorem of this section shows that the Omega point is the unique allocation rule that satisfies the aforementioned five properties.

Theorem 4.6 *An allocation rule on $(N, D, Z) \in \Upsilon$ satisfies the properties EF, NE, IBC, TI and ACP if and only if it coincides with the Omega point.*

Proof (If) The *if* part of the proof is direct from Proposition 4.4.
(Only if) Let γ be an allocation rule. The case where the number of essential players is zero, namely $|\mathcal{E}| = 0$, follows from Proposition 4.5. Then, it remains to prove the case when $|\mathcal{E}| \geq 1$.In this case, we know that $D = D_1 + D_2 + \cdots + D_T$ where D_t is (see (3)):

$$D_t = \begin{pmatrix} 0 \dots 0 \; d_t^1 \; 0 \dots 0 \\ 0 \dots 0 \; d_t^2 \; 0 \dots 0 \\ \vdots \dots 0 \; \vdots \; 0 \dots \vdots \\ 0 \dots 0 \; d_t^n \; 0 \dots 0 \end{pmatrix}.$$

Then for all $t \in T$, (N, D_t, Z) is a PI-situation with D_t an integer matrix. This implies that (N, D_t, Z) belongs to Υ. Therefore, for any $t \in T$, the Omega point for (N, D_t, Z) is $(\omega_i(N, D_t, Z))_{i=1,\dots,n}$:

By NE, $\gamma(N, D_t, Z) \neq \varnothing$. for each situation (N, D_t, Z) we have two cases:

- $\left|\mathcal{E}^t\right| = 0$, then by Proposition 4.5 $\gamma(N, D_t, Z) = \omega(N, D_t, Z) = Owen(N, D_t, Z)$.
- $\mathcal{E}^t = \{k\}$. Take $u \in \gamma(N, D_t, Z)$, by IBC $u_i \leq y_t^*(N \backslash \{k\}) d_t^i$ for all $i \in N \backslash \{k\}$ and by TY $u_{N\backslash\{k\}} = y_t^*(N\backslash\{k\}) d_t^{N\backslash\{k\}}$. Hence for all $i \in N\backslash\{k\}$ $u_i = y_t^*(N\backslash\{k\}) d_t^i = y_t^*(N) d_t^i + q_t(\mathcal{E}^t, i) = \omega_i(N, D_t, Z)$. Finally, by EF $u_k = c(N) - c(N\backslash\{k\}) = y_t^*(N) d_t^N - y_t^*(N\backslash\{k\}) d_t^{N\backslash\{k\}} = y_t^*(N) d_t^k - \left(y_t^*(N\backslash\{k\}) - y_t^*(N)\right) d_t^{N\backslash\{k\}} = y_t^*(N) d_t^k - \sum_{j \in F^t} q_t(k, j) = \omega_k(N, D_t, Z)$.

Therefore, if $x \in \gamma(N, D, Z)$ by ACP one has that $x = z_1 + \cdots + z_t$ with $z_t \in \gamma(N, D_t, Z)$ for all $t \in T$, and so

$$x = \sum_{t=1}^{T} z_t = \sum_{t \in T / \left|\mathcal{E}^t\right| \leq 1} \omega(N, D_t, Z) + \sum_{t \in T / \left|\mathcal{E}^t\right| \geq 2} Owen(N, D_t, Z) = \omega(N, D, Z).$$

The above equation implies that $\gamma(N, D, Z) = \omega(N, D, Z)$. □

Finally, we prove that all the properties used in Theorem 4.6 are logically independent. That is, the characterization of the Omega point is tight in the sense that no property is redundant.

Example 4.7 Let γ be a solution rule defined on Υ as

$$\gamma(N, D, Z) := \begin{cases} \left(\frac{c^{(N,D,Z)}(N)}{2}, \frac{c^{(N,D,Z)}(N)}{2} \right), & (N, D, Z) \in \Upsilon^1 \\ \\ \omega(N, D, Z), & \text{otherwise,} \end{cases}$$

where

$$\Upsilon^1 := \{(N, D, Z) \in \Upsilon / |N| = 2, T = 2, \mathcal{E}^1 = \{1, 2\}, \mathcal{E}^2 = \varnothing\}.$$

$\gamma(N, D, Z)$ satisfies EF, NE, IBC and TI, but not ACP.

Example 4.8 Let γ be a solution rule defined on Υ as

$$\gamma(N, D, Z) := \begin{cases} \left(c^{(N,D,Z)}(N), 0\right), & (N, D, Z) \in \Upsilon^2 \\ \omega(N, D, Z), & \text{otherwise,} \end{cases}$$

where

$$\Upsilon^2 := \{(N, D, Z) \in \Upsilon / |N| = 2, T = 1, \mathcal{E} = \emptyset\}.$$

$\gamma(N, D, Z)$ satisfies EF, NE, ACP and TI, but not IBC.

Example 4.9 Let γ be a solution rule defined on Υ as

$$\gamma(N, D, Z) := Owen(N, D, Z)$$

$\gamma(N, D, Z)$ satisfies EF, NE, IBC, and ACP, but not TI.

Example 4.10 Let γ be a solution rule defined on Υ as

$$\gamma(N, D, Z) := \begin{cases} \left(c^{(N,D,Z)}(N \backslash \{\mathcal{E}^1\}), c^{(N,D,Z)}(N \backslash \{\mathcal{E}^1\})\right), & (N, D, Z) \in \Upsilon^3 \\ \omega(N, D, Z), & \text{otherwise,} \end{cases}$$

where

$$\Upsilon^3 := \{(N, D, Z) \in \Upsilon / |N| = 2, T = 1, |\mathcal{E}^1| = 1\}.$$

$\gamma(N, D, Z)$ satisfies NE, IBC, TI and ACP, but not EF.

Example 4.11 Let γ be a solution rule defined on Υ as

$$\gamma(N, D, Z) := \emptyset.$$

$\gamma(N, D, Z)$ satisfies EF, ACP, IBC, and TI. but not NE.

5 Quid Pro Quo allocations

As we already mentioned, the Omega point can be considered the natural aspiration of the essential players to achieve the biggest cost reduction while the Owen point reflects their altruistic character. We combine both extreme characteristics and define the λ-agreement $a(\lambda) := \lambda\omega + (1 - \lambda)\, o$ with $\lambda \in [0, 1]$, as the convex linear combination of the Owen point and the Omega point. The parameter λ represents here the weight given to individual behavior, by those players who want to maximize their cost reduction, compared to altruistic behavior (by $1 - \lambda$), which benefits the other players.

The set of all the above agreements is called Quid Pro Quo allocation set.

Definition 5.1 (Quid Pro Quo allocation set) Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. We define the Quid pro quo allocation set as follows:

$$QPQ(N, c) := \{a(\lambda) \text{ such that } \lambda \in [0, 1]\}.$$

The Quid Pro Quo allocation set, henceforth QPQ-set, is a parametric family of core-allocations. That is, $QPQ(N, c) \subseteq Core(N, c)$.

The following example illustrate the wealth of the QPQ-set of a PI-situation with multiple essential players.

Example 5.2 Let us consider a PI-situation with four players in four periods (Table 2):

The characteristic function of the corresponding PI-game is described below (Table 3):

Here, $y^*(N) = (1, 1, 1, 1)$ and the Owen point is $o = (7, 7, 7, 6)$. Moreover, $\mathcal{E} = N$, because of each player is essential just in one period. For example in period 1, $\mathcal{E}^1 = \{1\}$ and $F^1 = \{2, 3, 4\}$. In addition, $q_1(1, j) = 2$ for $j \in F^1$, $Q^1 = (-6, 2, 2, 2)$, $Q^2 = (1, -3, 1, 1)$, $Q^3 = (2, 1, -4, 1)$ and $Q^4 = (2, 2, 2, -6)$.

It is easy to check that $Q = (-1, 2, 1, -2)$ that is, players 1 and 4 are interested in improving the Owen point, and they would prefer the Omega point. However, players 2 and 3, still being essential, get some benefit with the Owen point's and they would prefer to keep on it.

On the other hand, here the omega point is $\omega = (6, 9, 8, 4)$ and the QPQ-set is given by:

$$QPQ(N, c) := \{(7 - \lambda, 7 + 2\lambda, 7 + \lambda, 6 - 2\lambda) \text{ such that } \lambda \in [0, 1]\}$$

Table 2 PI-situation with 4 players and 3-periods in Example 5.2

	Demand				Production				Inventory			Backlogging		
P1	2	1	2	2	1	2	2	2	1	1	1	2	2	2
P2	2	2	1	2	2	1	2	2	1	1	1	2	2	2
P3	2	1	2	2	2	2	1	2	1	1	1	2	2	2
P4	2	1	1	2	2	2	2	1	1	1	1	2	2	2

Table 3 PI-game with 4 players in Example 5.2

	d_1^S	d_2^S	d_3^S	d_4^S	p_1^S	p_2^S	p_3^S	p_4^S	h_1^S	h_2^S	h_3^S	b_1^S	b_2^S	b_3^S	c
{1}	2	1	2	2	1	2	2	2	1	1	1	2	2	2	12
{2}	2	2	1	2	2	1	2	2	1	1	1	2	2	2	12
{3}	2	1	2	2	2	2	1	2	1	1	1	2	2	2	12
{4}	2	1	1	2	2	2	2	1	1	1	1	2	2	2	10
{1, 2}	4	3	3	4	1	1	2	2	1	1	1	2	2	2	21
{1, 3}	4	2	4	4	1	2	1	2	1	1	1	2	2	2	20
{1, 4}	4	2	3	4	1	2	2	1	1	1	1	2	2	2	18
{2, 3}	4	3	3	4	2	1	1	2	1	1	1	2	2	2	22
{2, 4}	4	3	2	4	2	1	2	1	1	1	1	2	2	2	19
{3, 4}	4	2	3	4	2	2	1	1	1	1	1	2	2	2	19
{1, 2, 3}	6	4	5	6	1	1	1	2	1	1	1	2	2	2	27
{1, 2, 4}	6	4	5	6	1	1	2	1	1	1	1	2	2	2	24
{1, 3, 4}	6	3	5	6	1	2	1	1	1	1	1	2	2	2	23
{2, 3, 4}	6	4	4	6	2	1	1	1	1	1	1	2	2	2	26
{1, 2, 3, 4}	8	5	6	8	1	1	1	1	1	1	1	2	2	2	27

If we consider the same weight for both individual and altruistic behaviors, we get the Shapley, which also matches the Nucleolus. That is, for $\lambda = \frac{1}{2}$ the Shapley value and Nucleolus coincides and both are equal to $\left(\frac{13}{2}, 8, \frac{15}{2}, 5\right)$.

At this point we wonder whether this coincidence always holds for every PI-game. The answer is no, in general, as example 5.5 reveals.

The main result of this section shows that, if no player can get a cost reduction in any coalition without an essential player, then the equal agreement, $a\left(\frac{1}{2}\right)$, coincides with the Shapley value and the Nucleolus. In some sense, it is a Solomonic agreement between the players who demand cost reductions (individual behaviour) and those who do not (altruistic behaviour). For that, we call $a\left(\frac{1}{2}\right)$ Solomonic allocation and denote it $\varsigma(N, c)$.

Proposition 5.3 *Let $(N, D, Z) \in \Upsilon$ and (N, c) be the corresponding PI-game. Assume that for each $t = 1, \ldots, T$ the following conditions are simultaneously fulfilled:*

(i) $\left|\mathcal{E}^t\right| \leq 1$,
(ii) $y_t^*(\mathcal{E}^t) = y_t^*(N)$ *if* $\mathcal{E}^t \neq \phi$,
(iii) $y_t^*(N \setminus \mathcal{E}^t) = y_t^*(\{i\})$ *for all* $i \in N \setminus \mathcal{E}^t$.
Then, $\varsigma(N, c) = \phi(N, c) = \eta(N, c)$.

Proof Consider $(N, D_t, Z) \in \Upsilon$ and (N, c^t) be the corresponding PI-game, with, (only period t has demand)

$$D_t = \left(d^{ip}\right)_{\substack{i=1,\ldots,n \\ p=1,\ldots,T}}, \ d^{ip} = \begin{cases} d_p^i & \text{if } t = p, \\ 0 & \text{otherwise.} \end{cases}$$

We will denote to simplify notation $o(N, D_t, Z)$ and $\omega(N, D_t, Z)$ as o^t and ω^t, respectively. By (i) we consider only two cases:

- If $\left|\mathcal{E}^t\right| = 0$ then $\omega^t = o^t = Core(N, c^t) = \eta(N, c^t)$ since the Nucleolus always belongs to the core of a game. Moreover, because of the condition (iii) $y_t^*(N) = y_t^*(\{i\})$ for all $i \in N$, then $c^t(S) = o_S^t$ for all $S \subseteq N$, It is easy to verify that all players are dummy players then $\phi(N, c^t) = o^t$.
- If $\left|\mathcal{E}^t\right| = 1$, $(\mathcal{E}^t = \{k\})$. Note that if $k \in S$ then $c^t(S \cup \{i\}) - c^t(S) = y_t^*(N) d_t^i$, otherwise $(k \notin S)$ then by condition (iii) $c^t(S \cup \{i\}) - c^t(S) = y_t^*(S \cup \{i\}) d_t^{S \cup \{i\}} - y_t^*(S) d_t^S = y_t^*(N \setminus \mathcal{E}^t) d_t^i$ for all $i \in N \setminus \mathcal{E}^t$. If $i \in N \setminus \mathcal{E}^t$ then,

$$\begin{aligned}
\phi_i(N, c^t) = & \sum_{S \subseteq N \setminus \{i\}} \gamma(S) \cdot \left[c^t(S \cup \{i\}) - c^t(S)\right] = \\
& \sum_{S \subseteq N \setminus \{i\} / k \in S} \gamma(S) \cdot \left[c^t(S \cup \{i\}) - c^t(S)\right] \\
& + \sum_{S \subseteq N \setminus \{i\} / k \notin S} \gamma(S) \cdot \left[c^t(S \cup \{i\}) - c^t(S)\right] \\
= & \sum_{S \subseteq N \setminus \{i\} / k \in S} \gamma(S) \cdot y_t^*(N) d_t^i + \sum_{S \subseteq N \setminus \{i\} / k \notin S} \gamma(S) \cdot y_t^*(N \setminus \mathcal{E}^t) d_t^i \\
= & \, y_t^*(N) d_t^i \cdot \left(\sum_{S \subseteq N \setminus \{i\} / k \in S} \gamma(S) \right)
\end{aligned}$$

$$+y_t^*(N\backslash\mathcal{E}^t)d_t^i\cdot\left(\sum_{S\subseteq N\setminus\{i\}/k\notin S}\gamma(S)\right)$$
$$=\frac{1}{2}\cdot y_t^*(N)d_t^i+\frac{1}{2}\cdot y_t^*(N\backslash\mathcal{E}^t)d_t^i$$
$$=\frac{1}{2}\cdot y_t^*(N)d_t^i+\frac{1}{2}\cdot\left(y_t^*(N)d_t^i+\left(y_t^*(N\backslash\{\mathcal{E}^t\})-y_t^*(N)\right)\cdot d_t^i\right)$$
$$=\frac{1}{2}\cdot o_i^t+\frac{1}{2}\cdot\omega_i^t$$

By efficiency of Shapley value $\phi_{\mathcal{E}^t}(N,c^t)=\frac{1}{2}\cdot o_{\mathcal{E}^t}^t+\frac{1}{2}\cdot\omega_{\mathcal{E}^t}^t$. Moreover, Shapley value satisfies additivity property, thus for all player $i\in N$

$$\phi_i(N,c)=\sum_{t=1}^T\phi_i(N,c^t)=\sum_{t=1}^T\left(\frac{1}{2}\cdot o_i^t+\frac{1}{2}\cdot\omega_i^t\right)$$
$$=\frac{1}{2}\cdot\sum_{t=1}^T o_i^t+\frac{1}{2}\cdot\sum_{t=1}^T\omega_i^t$$
$$=\frac{1}{2}\cdot o_i(N,D,Z)+\frac{1}{2}\cdot\omega_i(N,D,Z)$$

since the Owen point is additive for the demands [demonstrated in Guardiola et al. (2008)] and $\omega_i(N,D_t,Z)=\omega_i^t(N,D,Z)$. Hence, $\varsigma(N,c)=\phi(N,c)$. Now, we will prove that the Shapley value coincides with the Nucleolus. As we have seen previously if the properties (i), (ii) and (iii) are satisfied for a period $t=1,\ldots,T$ and for each $i\in N$ and for all $S\subseteq N\backslash\{i\}$

$$\Delta_i^t(S):=c^t(S\cup\{i\})-c^t(S)=\begin{cases}y_t^*(N)d_t^i & if\ \mathcal{E}^t\in S \text{ and } i\notin\mathcal{E}^t\\ y_t^*(N\backslash\mathcal{E}^t)d_t^i & if\ \mathcal{E}^t\notin S \text{ and } i\notin\mathcal{E}^t\\ y_t^*(N)d_t^i-y_t^*(N\backslash\mathcal{E}^t)d_t^i & if\ \quad i\in\mathcal{E}^t\end{cases}$$

similarly we get that

$$\Delta_i^t(N\backslash(S\cup\{i\}))=\begin{cases}y_t^*(N\backslash\mathcal{E}^t)d_t^i & if\ \mathcal{E}^t\in S \text{ and } i\notin\mathcal{E}^t\\ y_t^*(N)d_t^i & if\ \mathcal{E}^t\notin S \text{ and } i\notin\mathcal{E}^t\\ y_t^*(N)d_t^i-y_t^*(N\backslash\mathcal{E}^t)d_t^i & if\ \quad i\in\mathcal{E}^t\end{cases}$$

Hence, $\Delta_i^t(S)+\Delta_i^t(N\backslash(S\cup\{i\}))=y_t^*(N)d_t^i+y_t^*(N\backslash\mathcal{E}^t)d_t^i$ if $i\in N\backslash\mathcal{E}^t$ for all $S\subseteq N\backslash\{i\}$ and $\Delta_i^t(S)+\Delta_i^t(N\backslash(S\cup\{i\}))=2\cdot\left(y_t^*(N)d_t^i-y_t^*(N\backslash\mathcal{E}^t)d_t^i\right)$ if $i\in\mathcal{E}^t$ for all $S\subseteq N\backslash\mathcal{E}^t$. We consider $\Delta_i(S):=\sum_{t=1}^T\Delta_i^t(S)$ for each $i\in N$ and for all $S\subseteq N\backslash\{i\}$. Thus $\Delta_i(S)+\Delta_i(N\backslash(S\cup\{i\}))$ is a constant for all $S\subseteq N\backslash\{i\}$ and for all $i\in N$. Then (N,c) is a PS-game and $\varsigma(N,c)=\phi(N,c)=\eta(N,c)$.

The reader may notice that for those situations in which the properties (i) , (ii) and (iii) hold and, in addition, $Q=0$ (i.e., $o=\omega$), then $QPQ(N,c)=\{o(N,D,Z)\}=\{\phi(N,c)\}=\{\eta(N,c)\}$. Otherwise, the core is larger.

Finally, we analyze the relationships between conditions (i), (ii), (iii) and concavity of PI-games.

Proposition 5.4 *Let $(N,D,Z)\in\Upsilon$ and (N,c) be the corresponding PI-game. If for each $t=1,\ldots,T$ conditions (i), (ii) and (ii) are fulfilled simultaneously the (N,c) is concave.*

Proof Consider $(N, D_t, Z) \in \Upsilon$ and (N, c^t) be the corresponding PI-game,

(a) If $|\mathcal{E}^t| = 0$ then $y_t^*(\{i\}) = y_t^*(N)$ for all $i \in N$ henceforth $c^t(S) - c^t(S\backslash\{i\}) = y_t^*(N)d_t^i$ for all $i \in N$ and for all $S \subseteq N$. Hence (N, c^t) is concave.

(b) If $|\mathcal{E}^t| = 1$, let say $\mathcal{E}^t = \{k\}$. Then two cases can be distinguished:

(b1) $k \in S \subseteq T \subset N$ then $c^t(S) - c^t(S\backslash\{i\}) = y_t^*(N)d_t^i = c^t(T) - c^t(T\backslash\{i\})$ for all $i \in N\backslash\{k\}$. Finally

$$c^t(S) - c^t(S\backslash\{k\}) \geq c^t(T) - c^t(T\backslash\{k\});$$
$$y_t^*(N)d_t^S - y_t^*(N\backslash\{k\})d_t^{S\backslash\{k\}} \geq y_t^*(N)d_t^T - y_t^*(N\backslash\{k\})d_t^{T\backslash\{k\}};$$
$$y_t^*(N\backslash\{k\})d_t^{T\backslash S} \geq y_t^*(N)d_t^{T\backslash S}.$$

It is true since $y_t^*(S) \geq y_t^*(R)$ for all $S \subseteq R \subseteq N$ and all $t \in \{1, \ldots, T\}$.

(b2) $k \notin S$ and $k \in T$. By condition (iii) $c^t(S) - c^t(S\backslash\{i\}) = y_t^*(N\backslash\{k\})d_t^i \geq c^t(T) - c^t(T\backslash\{i\})$ since if $k \in T$ is satisfied $c^t(T) - c^t(T\backslash\{i\}) = y_t^*(N)d_t^i$ and if $k \notin T$ we have that $c^t(T) - c^t(T\backslash\{i\}) = y_t^*(N\backslash\{k\})d_t^i$.

Finally, by additivity property of PI-games with respect to periods [see Guardiola et al. (2008)] (N, c) is concave. □

Next example shows that conditions (i), (ii), (iii), although necessaries, are no sufficient for concavity.

Example 5.5 Let us consider a PI-situation with three players in three periods: Table 4.

Using those data one can obtain the cooperative game with characteristic function described below: Table 5.

It is easy to check that the above game is concave, but condition (iii) does not hold. Indeed, for the first period, $\mathcal{E}^1 = \{1\}$ but $y_1^*(\{2, 3\}) = 2 < 3 = y_1^*(\{3\})$.

Moreover, the Nucleolus $\eta(N, c) = \left(\frac{70}{3}, \frac{85}{3}, \frac{100}{3}\right)$ is lightly different from the Shapley value, $\phi(N, c) = \left(\frac{125}{6}, \frac{155}{6}, \frac{115}{3}\right)$.

Table 4 PI-situation with 3 players and periods in Example 5.5

	Demand			Production			Inventory		Backlogging	
P1	10	10	10	1	2	3	1	2	1	1
P2	10	10	10	2	1	3	1	2	1	1
P3	10	10	10	3	3	1	1	2	2	2

Table 5 PI-game with 3 playes in Example 5.5

	d_1^S	d_2^S	d_3^S	p_1^S	p_2^S	p_3^S	h_1^S	h_2^S	b_1^S	b_2^S	c
{1}	10	10	5	1	2	3	1	2	1	1	45
{2}	10	10	10	2	1	3	1	1	1	1	50
{3}	10	10	10	3	3	1	1	2	2	2	70
{1, 2}	20	20	15	1	1	3	1	1	1	1	70
{1, 3}	20	20	15	1	2	1	1	2	1	1	75
{2, 3}	20	20	20	2	1	1	1	1	1	1	80
{1, 2, 3}	30	30	25	1	1	1	1	1	1	1	85

Finally, $o = (30, 30, 25)$, $\omega = (25, 30, 30)$ and so the Solomonic allocation is $\varsigma(N, c) = \left(\frac{55}{2}, 30, \frac{55}{2}\right)$.

6 Concluding remarks

This paper completes the study of the PI-games presented in Guardiola et al. (2008, 2009). Those two papers proposed the Owen point as a natural core-allocation, which does not pay attention to the role that essential players play in reducing the costs of their followers. In that sense, essential players could consider the Owen point as an altruistic core-allocation. However, the core was not studied in depth there.

Here we have analyzed carefully the core structure of PI-games, and we have realized that the number of extreme point of its core is exponential in the number of players. Then, we have proposed a new core-allocation, the Omega point, that compensates the essential players for their role in reducing the costs of their followers. Based on the Owen and Omega points we have defined the QPQ-set. Since every QPQ allocation is a convex combination of the Owen and the Omega points, we have paid special attention to the equally weighted QPQ allocation, the Solomonic allocation. Finally, we have provided some necessary conditions for the coincidence of the latter with the Shapley value and the Nucleolus. A further extension of PI-games to a more general setting with set-up costs can be found in Guardiola et al. (2021).

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